LIMIT VALUES OF THE NON-ACYCLIC REIDEMEISTER TORSION FOR KNOTS

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Abstract. We consider the Reidemeister torsion associated with $\mathrm{SL}_2(\mathbb{C})$-representations of a knot group. A bifurcation point in the $\mathrm{SL}_2(\mathbb{C})$-character variety of a knot group is a character which is given by both an abelian $\mathrm{SL}_2(\mathbb{C})$-representation and a non-abelian one. We show that there exist limits of the non-acyclic Reidemeister torsion at bifurcation points and the limits are expressed by using the derivation of the Alexander polynomial of the knot in this paper.

1. Introduction

The Reidemeister torsion is an invariant of a CW-complex and a representation of its fundamental group. For a knot exterior and an abelian representation, the Reidemeister torsion is essentially equal to the Alexander polynomial, see Milnor [15, 17] and Turaev [20]. In the case of a non-abelian representation, the Reidemeister torsion is related to the theory of the twisted Alexander invariant, see Kirk and Livingston [11], Kitano [12], Lin [14] and Wada [22].

The Reidemeister torsion is invariant under taking conjugation of a representation. And in the case of knot exteriors the Reidemeister torsion may be regarded as a function on a space corresponding to a suitable quotient of the $\mathrm{SL}_2(\mathbb{C})$-representations of the knot group by conjugation, we can find that this point of view was introduced in Porti [19]. Following Morgan and Shalen [18], we consider the $\mathrm{SL}_2(\mathbb{C})$-character variety of the knot group as a suitable quotient. In general, the $\mathrm{SL}_2(\mathbb{C})$-character variety of a knot group has many components. These components are roughly classified into two types. One consists of the characters of abelian representations. The other consists of the characters of non-abelian representations. We respectively call these sets the *abelian part* and the *non-abelian part* of the character variety. It is known that the abelian part intersects with the non-abelian part. These intersection points are called *bifurcation points*. The purpose of this paper is to show that the Reidemeister torsion of non-abelian representations is given by using the Alexander polynomials at a bifurcation point as follows.

Let K be a knot in a homology three sphere. A bifurcation point of the $\mathrm{SL}_2(\mathbb{C})$-character variety of K corresponds to a root of the Alexander polynomial, see Burde [1] and Klassen [13]. In particular, the bifurcation point corresponding to a simple root of the Alexander polynomial is a smooth point of the $\mathrm{SL}_2(\mathbb{C})$-character variety (see Heusener, Porti and Suárez [10]). We can construct a function on each of the abelian and non-abelian part of the character variety by using the Reidemeister torsion. The function on the abelian part is given by the Reidemeister torsion for abelian representations. In fact, this function is expressed by using the Alexander polynomial of K and it has zeros at bifurcation points (see Milnor [15, 17] and

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Turaev [20]). The other function on the non-abelian part is given by the *non-acyclic* Reidemeister torsion for non-abelian representations, Dubois [6, 7], Porti [19] and Yamaguchi [23] deal with Reidemeister torsion in such a light. Though the function on the non-abelian part is partially defined and it is not defined on bifurcation points, we can consider limits of the non-acyclic Reidemeister torsion at bifurcation points.

We will show that if a bifurcation point corresponds to a simple root of the Alexander polynomial of K, then there exists the limit of the non-acyclic Reidemeister torsion at the bifurcation point and its limit is expressed as the differential coefficient of the function defined on the abelian part at this point (Theorem 3.2.1).

This fact had been first conjectured by Dubois and Kashaev. The author proved it for a knot in S^3 at first. Dubois pointed out that the proof may be extended to a knot in a homology three sphere. This theorem is applied in the paper of Dubois and Kashaev [8].

This paper is organized as follows. In Section 2, we recall the needed notions of the $\mathrm{SL}_2(\mathbb{C})$-character variety of a knot group and the Reidemeister torsion for knot exteriors. In Section 3, we prove that limits of the non-acyclic Reidemeister torsion of a knot exterior at bifurcation points are obtained from the derivation of the Alexander polynomial of the knot. We discuss the existences of limits of the non-acyclic Reidemeister torsion in Subsection 3.1. We give a formula of these limits in Subsection 3.2. This formula implies that some property, called *λ-regularity*, which holds on irreducible characters near a bifurcation point can be extended to the bifurcation point. This is shown in Section 4.

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2. Preliminary

2.1. **Review on bifurcation points.** Let K be a knot in a homology three sphere M, M_K its exterior and $R(\pi_1(M_K), \mathrm{SL}_2(\mathbb{C}))$ denote the set of $\mathrm{SL}_2(\mathbb{C})$-representations of $\pi_1(M_K)$.

A representation ρ is called *abelian* if its image $\rho(\pi_1(M_K))$ is an abelian subgroup of $\mathrm{SL}_2(\mathbb{C})$. A representation ρ is called *reducible* if there exists a proper subspace U of $\mathbb{C}^2$ such that $\rho(\gamma)(U) \subset U$ for any $\gamma \in \pi_1(M_K)$. A representation ρ is called *irreducible* if it is not reducible. We let $R^{irr}(\pi_1(M_K), \mathrm{SL}_2(\mathbb{C}))$ denote the set of irreducible ones. Note that all abelian representations are reducible but the converse is false in general.

Associated to the representation $\rho \in R(\pi_1(M_K), \mathrm{SL}_2(\mathbb{C}))$ is its *character* a map χ_ρ from $\pi_1(M_K)$ into $\mathbb{C}$, defined by $\chi_\rho(\gamma) = \mathrm{Tr}\,(\rho(\gamma))$. Following Morgan and Shalen [18], we will focus on the *character variety* which is the set of *characters* of $\mathrm{SL}_2(\mathbb{C})$-representations of $\pi_1(M_K)$. Let $X(M_K)$ denote the character variety of $\pi_1(M_K)$. In some sense, $X(M_K)$ is the "algebraic quotient" of $R(\pi_1(M_K), \mathrm{SL}_2(\mathbb{C}))$ by $\mathrm{PSL}_2(\mathbb{C})$ because the quotient $R(\pi_1(M_K), \mathrm{SL}_2(\mathbb{C}))/\mathrm{PSL}_2(\mathbb{C})$ is not Hausdorff in general. We let π denote the projection, $R(\pi_1(M_K), \mathrm{SL}_2(\mathbb{C})) \to X(M_K)$, defined by $\rho \mapsto \chi_\rho$. It is known that $R(\pi_1(M_K), \mathrm{SL}_2(\mathbb{C}))$ and $X(M_K)$ have the structure of complex algebraic affine sets and for each $\gamma \in \pi_1(M_K)$ the function $I_\gamma : X(M_K) \to \mathbb{C}, \chi_\rho \mapsto \mathrm{Tr}\,(\rho(\gamma))$ is a regular function. Two irreducible representations of $\pi_1(M_K)$ with the same character are conjugate by an element of $\mathrm{SL}_2(\mathbb{C})$ (see Culler and Shalen [4, Proposition 1.5.2]). Let $X^{irr}(M_K)$ denote $\pi(R^{irr}(\pi_1(M_K), \mathrm{SL}_2(\mathbb{C})))$.

The subsets $R^{irr}(\pi_1(M_K), \mathrm{SL}_2(\mathbb{C})) \subset R(\pi_1(M_K), \mathrm{SL}_2(\mathbb{C}))$ and $X^{irr}(M_K) \subset X(M_K)$ are Zariski-open. (For the details, see Morgan and Shalen [18].)

The character variety $X(M_K)$ has several components. Let $X^{ab}(M_K)$ be the image under π of the subset of abelian $\mathrm{SL}_2(\mathbb{C})$-representations of $\pi_1(M_K)$ and X^{nab} the image of the subset of non-abelian ones. We call $X^{ab}(M_K)$ (resp. $X^{nab}(M_K)$) the *abelian* (resp. *non-abelian*) part of $X(M_K)$.

Definition 2.1.1. If there exist intersection points between the abelian part $X^{ab}(M_K)$ and the non-abelian part $X^{nab}(M_K)$ in $X(M_K)$, then these intersection points are called *bifurcation points*.

It is well known that $\pi_1(M_K)/[\pi_1(M_K),\ \pi_1(M_K)] \cong H_1(M_K;\mathbb{Z}) \cong \mathbb{Z}$ is generated by the meridian μ of K.

Remark 2.1.2. In $\mathrm{SL}_2(\mathbb{C})$ there exist, up to conjugation, only two maximal abelian subgroups Hyp(=hyperbolic) and Para(=parabolic); they are given by

$$\mathrm{Hyp} := \left\{ \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}) \,\middle|\, c \in \mathbb{C}^* = \mathbb{C} \setminus \{0\} \right\},$$
$$\mathrm{Para} := \left\{ \pm \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}) \,\middle|\, \omega \in \mathbb{C} \right\}.$$

As a consequence, each abelian representation of $\pi_1(M_K)$ in $\mathrm{SL}_2(\mathbb{C})$ is conjugate either to

$$\varphi_z : \pi_1(M_K) \ni \mu \mapsto \begin{pmatrix} e^z & 0 \\ 0 & e^{-z} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$$

with $z \in \mathbb{C}$ if it is hyperbolic, or to a representation ρ with $\rho(\mu) = \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ if it is parabolic.

The non-abelian part $X^{nab}(M_K)$ includes the irreducible characters $X^{irr}(M_K)$. It is known that an element of $X^{irr}(M_K)$ is a smooth point in the complex affine variety $X(M_K)$, for example see Porti [19, Proposition 3.5]. We focus on the bifurcation points which are limits of paths in $X^{irr}(M_K)$. Such bifurcation points are related to roots of the Alexander polynomial $\Delta_K(t)$ of K. This is a well-known result of Burde [1], de Rham [5] if K is a knot in S^3.

Lemma 2.1.3 (Corollary 4.3 in Heusener–Porti–Suárez [10], Klassen [13])**.** *Let z_0 be a complex number. There is a reducible non-abelian representation ρ_{z_0} such that $\chi_{\rho_{z_0}} = \chi_{\varphi_{z_0}}$ if and only if $\Delta_K(e^{2z_0}) = 0$.*

It is also known that the following theorem holds.

Theorem 2.1.4 (Theorem 1.1 in Heusener–Porti–Suárez [10])**.** *Let z_0 be a complex number such that $\Delta_K(e^{2z_0}) = 0$ and ρ_{z_0} a reducible non-abelian representation such that $\chi_{\rho_{z_0}} = \chi_{\varphi_{z_0}}$. If e^{2z_0} is a simple root of $\Delta_K(t)$, then the representation ρ_{z_0} is the limit of a sequence of irreducible ones. More precisely, ρ_{z_0} is a smooth point of the $\mathrm{SL}_2(\mathbb{C})$-representation variety of $\pi_1(M_K)$; it is contained in a unique irreducible four-dimensional component of the $\mathrm{SL}_2(\mathbb{C})$-representation variety.*

Heusener, Porti and Suárez also showed that the character of ρ_{z_0} is a smooth point of the $\mathrm{SL}_2(\mathbb{C})$-character variety $X(M_K)$ (see Theorem 1.2 in Heusener–Porti–Suárez [10]).

We will consider bifurcation points corresponding to simple roots of the Alexander polynomial $\Delta_K(t)$. These bifurcation points are limits of paths in $X^{irr}(M_K)$.

2.2. **Review on the Reidemeister torsion.**

Torsion of a chain complex. Let $C_* = (0 \to C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} C_0 \to 0)$ be a chain complex of finite dimensional vector spaces over $\mathbb{C}$. Choose a basis $\mathbf{c}^i$ for C_i and a basis $\mathbf{h}^i$ for the i-th homology group $H_i = H_i(C_*)$. The torsion of C_* with respect to these choice of bases is defined as follows.

Let $\mathbf{b}^i$ be a sequence of vectors in C_i such that $d_i(\mathbf{b}^i)$ is a basis of $B_{i-1} = \operatorname{im}(d_i \colon C_i \to C_{i-1})$ and let $\widetilde{\mathbf{h}}^i$ denote a lift of $\mathbf{h}^i$ in $Z_i = \ker(d_i \colon C_i \to C_{i-1})$. The set of vectors $d_{i+1}(\mathbf{b}^{i+1})\widetilde{\mathbf{h}}^i\mathbf{b}^i$ is a basis of C_i. Let $[d_{i+1}(\mathbf{b}^{i+1})\widetilde{\mathbf{h}}^i\mathbf{b}^i/\mathbf{c}^i] \in \mathbb{C}^*$ denote the determinant of the transition matrix between those bases (the entries of this matrix are coordinates of vectors in $d_{i+1}(\mathbf{b}^{i+1})\widetilde{\mathbf{h}}^i\mathbf{b}^i$ with respect to $\mathbf{c}^i$). The *sign-determined Reidemeister torsion* of C_* (with respect to the bases $\mathbf{c}^*$ and $\mathbf{h}^*$) is the following alternating product (see Turaev [20, Definition 3.1]):

$$\operatorname{Tor}(C_*, \mathbf{c}^*, \mathbf{h}^*) = (-1)^{|C_*|} \cdot \prod_{i=0}^{n} [d_{i+1}(\mathbf{b}^{i+1})\widetilde{\mathbf{h}}^i\mathbf{b}^i/\mathbf{c}^i]^{(-1)^{i+1}} \in \mathbb{C}^*. \tag{1}$$

Here

$$|C_*| = \sum_{k \geqslant 0} \alpha_k(C_*)\beta_k(C_*),$$

where $\alpha_i(C_*) = \sum_{k=0}^{i} \dim C_k$, $\beta_i(C_*) = \sum_{k=0}^{i} \dim H_k$.

The torsion $\operatorname{Tor}(C_*, \mathbf{c}^*, \mathbf{h}^*)$ does not depend on the choices of $\mathbf{b}^i$ and $\widetilde{\mathbf{h}}^i$. Further observe that if C_* is acyclic (i.e., if $H_i = 0$ for all i), then $|C_*| = 0$.

Torsion of a CW-complex. Let W be a finite CW-complex, V a finite dimensional vector space over $\mathbb{C}$ and ρ a homomorphism from $\pi_1(W)$ to $Aut(V)$. We define the local system of W to be

$$C_*(W; V_\rho) = V_\rho \otimes_{\mathbb{Z}[\pi_1(W)]} C_*(\widetilde{W}; \mathbb{Z}).$$

Here $C_*(\widetilde{W}; \mathbb{Z})$ is the complex of the universal cover $\widetilde{W}$ with integer coefficients. This space is in fact a left $\mathbb{Z}[\pi_1(W)]$-module (via the action of $\pi_1(W)$ on $\widetilde{W}$ as the covering group). And V_ρ denotes the right $\mathbb{Z}[\pi_1(W)]$-module via the homomorphism ρ, i.e., the action is given by $v \cdot \gamma = \rho(\gamma)^{-1}(v)$ for any $v \in V$ and $\gamma \in \pi_1(W)$. This chain complex $C_*(W; V_\rho)$ computes the homology of the local system. We let $H_*(W; V_\rho)$ denote this homology.

Let $\{e_1^{(i)}, \ldots, e_{n_i}^{(i)}\}$ be the set of i-dimensional cells of W. We lift them to the universal cover and we choose an arbitrary order and an arbitrary orientation for the cells $\left\{\tilde{e}_1^{(i)}, \ldots, \tilde{e}_{n_i}^{(i)}\right\}$. If $\mathcal{B} = \{\mathbf{f_1}, \ldots, \mathbf{f_m}\}$ is an orthonormal basis of V, where m is the dimension of V, then we consider the corresponding basis over $\mathbb{C}$

$$\mathbf{c}^i_{\mathcal{B}} = \left\{\mathbf{f_1} \otimes \tilde{e}_1^{(i)}, \ldots, \mathbf{f_m} \otimes \tilde{e}_1^{(i)}, \ldots, \mathbf{f_1} \otimes \tilde{e}_{n_i}^{(i)}, \ldots, \mathbf{f_m} \otimes \tilde{e}_{n_i}^{(i)}\right\}$$

of $C_i(W; V_\rho)$. Now choosing for each i a basis $\mathbf{h}^i$ for the homology group $H_i(W; V_\rho)$, we can compute

$$\operatorname{Tor}(C_*(W; V_\rho), \mathbf{c}^*_{\mathcal{B}}, \mathbf{h}^*) \in \mathbb{C}^*.$$

The cells $\{\tilde{e}_j^{(i)}\}_{0 \leqslant i \leqslant \dim W, 1 \leqslant j \leqslant n_i}$ are in one-to-one correspondence with the cells of W, their order and orientation induce an order and an orientation for the cells $\{e_j^{(i)}\}_{i,j}$, where $0 \leqslant i \leqslant \dim W$ and $1 \leqslant j \leqslant n_i$. Again, corresponding to these choices, we get a basis c^i over $\mathbb{R}$ for $C_i(W; \mathbb{R})$.

Choose an *homology orientation* of W, which is an orientation of the real vector space $H_*(W; \mathbb{R}) = \bigoplus_{i \geqslant 0} H_i(W; \mathbb{R})$. Let $\mathfrak{o}$ denote this chosen orientation. Provide each vector space $H_i(W; \mathbb{R})$ with a reference basis h^i such that the basis $\left\{h^0, \ldots, h^{\dim W}\right\}$ of $H_*(W; \mathbb{R})$ is positively oriented with respect to $\mathfrak{o}$. We set

$$\tau_0 = \operatorname{sgn}\left(\operatorname{Tor}(C_*(W; \mathbb{R}), c^*, h^*)\right) \in \{\pm 1\}.$$

We define the sign–determined Reidemeister torsion for (W, V_ρ) with respect to the homology basis $\mathbf{h}^*$ and to the homology orientation $\mathfrak{o}$ to be

$$\mathrm{TOR}(W; V_\rho, \mathbf{h}^*, \mathfrak{o}) = \tau_0 \cdot \mathrm{Tor}(C_*(W; V_\rho), \mathbf{c}^*_{\mathcal{B}}, \mathbf{h}^*) \in \mathbb{C}^*. \tag{2}$$

This definition only depends on the combinatorial class of W, the conjugacy class of ρ, the choice of $\mathbf{h}^*$ and the homology orientation $\mathfrak{o}$. It is independent of the orthonormal basis $\mathcal{B}$ of V, of the choice of the lifts $\tilde{e}^{(i)}_j$, and of the choice of the positively oriented basis of $H_*(W; \mathbb{R})$. Moreover, it is independent of the order and the orientation of the cells (because they appear twice).

Remark 2.2.1. If the Euler characteristic of W is zero, then we can use any basis of V in order to define $\mathrm{TOR}(W; V_\rho, \mathbf{h}^*, \mathfrak{o})$.

One can prove that TOR is invariant under cellular subdivision, homeomorphism and simple homotopy equivalences. In fact, all these important invariance properties hold with the sign $(-1)^{|C_*|}$ in (1), for details see Farber and Turaev [9, Lemma 3.3].

2.3. **Review on the non-acyclic Reidemeister torsion for knot exteriors.** This subsection is devoted to a detailed review of the constructions of the non-acyclic Reidemeister torsion which were made in Dubois [6] and Porti [19].

Let K be a knot in a homology three sphere M and M_K its exterior. We let ρ denote an $\mathrm{SL}_2(\mathbb{C})$-representation of $\pi_1(M_K)$ and Ad be the adjoint action of $\mathrm{SL}_2(\mathbb{C})$, i.e., $Ad : \mathrm{SL}_2(\mathbb{C}) \to Aut(\mathfrak{sl}_2(\mathbb{C}))$, $A \mapsto (Ad_A : x \mapsto AxA^{-1})$.

We define the local system $C_*(M_K; \mathfrak{sl}_2(\mathbb{C})_\rho)$ by

$$C_*(M_K; \mathfrak{sl}_2(\mathbb{C})_\rho) := \mathfrak{sl}_2(\mathbb{C})_\rho \otimes_{\mathbb{Z}[\pi_1(M_K)]} C_*(\widetilde{M}_K; \mathbb{Z})$$

where $\widetilde{M}_K$ is the universal cover of M_K and $\mathfrak{sl}_2(\mathbb{C})_\rho$ is the right $\mathbb{Z}[\pi_1(M_K)]$-module via the composition $Ad \circ \rho$, i.e., $v \cdot \gamma = Ad_{\rho(\gamma)^{-1}}(v)$ for any $v \in \mathfrak{sl}_2(\mathbb{C})$ and $\gamma \in \pi_1(M_K)$. We call this local system *the* $\mathfrak{sl}_2(\mathbb{C})_\rho$*-twisted chain complex* of M_K.

We let $H_*(M_K; \mathfrak{sl}_2(\mathbb{C})_\rho)$ denote the homology of this local system. It is known that $\dim_{\mathbb{C}} H_1(M_K; \mathfrak{sl}_2(\mathbb{C})_\rho)$ is equal to the dimension of the component of $X(M_K)$ which contains χ_ρ if ρ is irreducible. In particular, for an irreducible representation ρ, $C_*(M_K; \mathfrak{sl}_2(\mathbb{C})_\rho)$ is not acyclic since there are no 0-dimensional components of $X(M_K)$ (see Cooper–Culler–Gillet–Long–Shalen [3, Proposition 2.4]).

Canonical homology orientation of knot exteriors. We provide the exterior of K with its *canonical homology orientation* defined as follows (see Turaev [21, Section V.3]). We have

$$H_*(M_K; \mathbb{R}) = H_0(M_K; \mathbb{R}) \oplus H_1(M_K; \mathbb{R})$$

and we base this $\mathbb{R}$-vector space with $\{[pt], [\mu]\}$. Here $[pt]$ is the homology class of a point, and $[\mu]$ is the homology class of the meridian μ of K. This reference basis of $H_*(M_K; \mathbb{R})$ induces the so–called canonical homology orientation of M_K. We let $\mathfrak{o}$ denote the canonical homology orientation of M_K.

Regularity for representations. In this subsection we briefly review two notions of regularity (see Dubois [7] and Porti [19]). Let $K \subset M$ denote an oriented knot.

The meridian μ of K is supposed to be oriented according to the rule $\ell k(K, \mu) = +1$, while the preferred longitude λ is oriented according to the condition $\mathrm{int}(\mu, \lambda) = +1$. Here $\mathrm{int}(\cdot, \cdot)$ denotes the intersection form on ∂M_K.

We say that $\rho \in R^{\mathrm{irr}}(\pi_1(M_K), \mathrm{SL}_2(\mathbb{C}))$ is *regular* if $\dim_{\mathbb{C}} H_1(M_K; \mathfrak{sl}_2(\mathbb{C})_\rho) = 1$. This notion is invariant by conjugation and thus it is well-defined for irreducible characters. Note that for a regular representation ρ, we have

$$\dim_{\mathbb{C}} H_1(M_K; \mathfrak{sl}_2(\mathbb{C})_\rho) = 1,\ \dim_{\mathbb{C}} H_2(M_K; \mathfrak{sl}_2(\mathbb{C})_\rho) = 1 \text{ and } H_j(M_K; \mathfrak{sl}_2(\mathbb{C})_\rho) = 0$$

for all $j \neq 1, 2$ by Porti [19, Corollary 3.23]. Let γ be a simple closed unoriented curve in ∂M_K. Among irreducible representations we focus on the γ-regular ones. We say that a regular representation $\rho : \pi_1(M_K) \to \mathrm{SL}_2(\mathbb{C})$ is *γ-regular* (see Porti [19, Definition 3.21]), if

(1) the inclusion $\iota\colon \gamma \hookrightarrow M_K$ induces a *surjective* map
$$\iota_*\colon H_1(\gamma; \mathfrak{sl}_2(\mathbb{C})_\rho) \to H_1(M_K; \mathfrak{sl}_2(\mathbb{C})_\rho);$$
(2) if $\mathrm{Tr}\,(\rho(\pi_1(\partial M_K))) \subset \{\pm 2\}$, then $\rho(\gamma) \neq \pm \mathbf{1}$.

It is easy to see that this notion is invariant by conjugation. For $\chi \in X^{\mathrm{irr}}(M_K)$ the notion of γ-regularity is well-defined.

How to construct natural bases for the twisted homology. Let ρ be a regular $\mathrm{SL}_2(\mathbb{C})$-representation of $\pi_1(M_K)$ and fix a generator P^ρ of $H_0(\partial M_K; \mathfrak{sl}_2(\mathbb{C})_\rho)$ (i.e., the vector P^ρ in $\mathfrak{sl}_2(\mathbb{C})$ satisfies the condition that $Ad_{\rho(g)}(P^\rho) = P^\rho$ for all $g \in \pi_1(\partial M_K)$).

Suppose that M is oriented. The exterior of a knot is thus oriented and we know that it is bounded by a 2-dimensional torus. This boundary inherits an orientation by the convention *"the inward pointing normal vector in the last position"*. The usual inclusion $i\colon \partial M_K \to M_K$ induces (see Dubois [6, Lemma 5.2]) an isomorphism $i_*\colon H_2(\partial M_K; \mathfrak{sl}_2(\mathbb{C})_\rho) \to H_2(M_K; \mathfrak{sl}_2(\mathbb{C})_\rho)$. Moreover, one can prove that $H_2(\partial M_K; \mathfrak{sl}_2(\mathbb{C})_\rho) \cong H_2(\partial M_K; \mathbb{Z}) \otimes \mathbb{C}$ (see Dubois [6, Lemma 5.1]). More precisely, let $[\partial M_K] \in H_2(\partial M_K; \mathbb{Z})$ be the fundamental class induced by the orientation of ∂M_K, we have that $H_2(\partial M_K; \mathfrak{sl}_2(\mathbb{C})_\rho) = \mathbb{C}[P^\rho \otimes \widetilde{\partial M_K}]$.

The *reference generator* of $H_2(M_K; \mathfrak{sl}_2(\mathbb{C})_\rho)$ is defined by

$$h^\rho_{(2)} = i_*([P^\rho \otimes \widetilde{\partial M_K}]). \tag{3}$$

Let ρ be a λ-regular representation of $\pi_1(M_K)$. Then the *reference generator* of $H_1(M_K; \mathfrak{sl}_2(\mathbb{C})_\rho)$ is defined by

$$h^\rho_{(1)}(\lambda) = \iota_*([P^\rho \otimes \widetilde{\lambda}]). \tag{4}$$

Remark 2.3.1. The generator $h^\rho_{(1)}(\lambda)$ of $H_1(M_K; \mathfrak{sl}_2(\mathbb{C})_\rho)$ depends on the orientation of λ. If we change the orientation of the longitude λ in Equation (4), then the generator changes into its reverse.

Remark 2.3.2. Note that $H_i(M_K; \mathfrak{sl}_2(\mathbb{C})_\rho)$ is isomorphic to the dual space of the twisted cohomology $H^i(M_K; \mathfrak{sl}_2(\mathbb{C})_\rho)$. The reference elements defined in Equations (3) and (4) are dual from ones defined in Dubois [7, § 3.4].

The non-acyclic Reidemeister torsion for knot exteriors. Let $\rho\colon \pi_1(M_K) \to \mathrm{SL}_2(\mathbb{C})$ be a λ-regular representation. The *Reidemeister torsion* $\mathbb{T}^K_\lambda$ at ρ is defined to be

$$\mathbb{T}^K_\lambda(\rho) = \mathrm{TOR}\left(M_K; \mathfrak{sl}_2(\mathbb{C})_\rho, \{h^\rho_{(1)}(\lambda), h^\rho_{(2)}\}, \mathfrak{o}\right) \in \mathbb{C}^*. \tag{5}$$

It is an invariant of knots. Moreover, if ρ_1 and ρ_2 are two λ-regular representations which have the same character, then $\mathbb{T}^K_\lambda(\rho_1) = \mathbb{T}^K_\lambda(\rho_2)$. Thus the Reidemeister torsion $\mathbb{T}^K_\lambda$ defines a map on the set $X^{\mathrm{irr}}_\lambda(M_K) = \{\chi \in X^{\mathrm{irr}}(M_K) \mid \chi \text{ is } \lambda\text{-regular}\}$ of λ-regular characters.

Remark 2.3.3. The Reidemeister torsion $\mathbb{T}^K_\lambda(\rho)$ defined in Equation (5) is exactly the inverse of the one considered in Dubois [7].

2.4. **Review on the acyclic Reidemeister torsion for knot exteriors.** We review the results of the Reidemeister torsion for acyclic local systems of knot exteriors in this section. Let K be a knot in a homology three sphere M and M_K its exterior.

The acyclic Reidemeister torsion of a knot exterior for abelian representations. Let ψ_z be a homomorphism from $\pi_1(M_K)$ to $\mathbb{C}^*$ such that $\psi_z(\mu) = e^z$ where z is a complex number and μ is the meridian of K. We let $C_*(M_K;\mathbb{C}_{\psi_z})$ denote the following local system:

$$\mathbb{C}_{\psi_z} \otimes_{\mathbb{Z}[\pi_1(M_K)]} C_*(\widetilde{M}_K;\mathbb{Z})$$

where $\widetilde{M}_K$ is the universal cover of M_K and $\mathbb{C}_{\psi_z}$ is a right $\mathbb{Z}[\pi_1(M_K)]$-module via the homomorphism ψ, i.e., $w \cdot \gamma = \psi_z(\gamma)^{-1}w$ for any $w \in \mathbb{C}$ and $\gamma \in \pi_1(M_K)$.

It is known that the torsion of $C_*(M_K;\mathbb{C}_{\psi_z})$ can be obtained from the normalized Alexander polynomial $\Delta_K(t)$ of K as follows.

Theorem 2.4.1 (Corollary 11.9 of Turaev [20])**.** *If z is a complex number such that $\Delta_K(e^z) \neq 0$, then the complex $C_*(M_K;\mathbb{C}_{\psi_z})$ is acyclic and* $\mathrm{Tor}(C_*(M_K;\mathbb{C}_{\psi_z}), \mathbf{c}^*_{\mathcal{B}})$ *is equal to*

$$\epsilon \cdot e^{nz/2}\frac{\Delta_K(e^z)}{e^{z/2}-e^{-z/2}}$$

where $\epsilon \in \{\pm 1\}$, n is some integer and $\mathcal{B}$ is a basis of the Lie algebra of $\mathbb{C}^*$, i.e., some non-zero element in $\mathbb{C}$.

We can regard the following function on $X^{ab}(M_K)$ as the Reidemeister torsion.

$$X^{ab}(M_K) \ni \chi_{\varphi_z} \mapsto \frac{\Delta_K(e^{2z})}{e^z - e^{-z}} \in \mathbb{C}.$$

The acyclic Reidemeister torsion of a knot exterior for $\mathrm{SL}_2(\mathbb{C})$*-representations.* Let α be the abelianization homomorphism of $\pi_1(M_K)$ which send the meridian μ to t. Let ρ be an $\mathrm{SL}_2(\mathbb{C})$-representation of $\pi_1(M_K)$. We let $C_*(M_K;\mathbb{C}(t)\otimes\mathfrak{sl}_2(\mathbb{C})_\rho)$ denote the following local system:

$$(\mathbb{C}(t)\otimes\mathfrak{sl}_2(\mathbb{C})_\rho) \otimes_{\mathbb{Z}[\pi_1(M_K)]} C_*(\widetilde{M}_K;\mathbb{Z})$$

where $\widetilde{M}_K$ is the universal cover of M_K and $\mathbb{C}(t)\otimes\mathfrak{sl}_2(\mathbb{C})_\rho$ is a right $\mathbb{Z}[\pi_1(M_K)]$-module via the action $\alpha\otimes(Ad\circ\rho)$, i.e., $(f(t)\otimes v)\cdot\gamma = f(t)t^{\alpha(\gamma)}\otimes Ad_{\rho(\gamma)^{-1}}(v)$ for any $f(t)\in\mathbb{C}(t)$, $v\in\mathfrak{sl}_2(\mathbb{C})$ and $\gamma\in\pi_1(M_K)$. For simplicity of notation, we let $\widetilde{\mathfrak{sl}}_2(\mathbb{C})_\rho$ stand for $\mathbb{C}(t)\otimes\mathfrak{sl}_2(\mathbb{C})_\rho$.

The following proposition holds for this chain complex.

Proposition 2.4.2 (Proposition 3.1.1 in Yamaguchi [23])**.** *If an* $\mathrm{SL}_2(\mathbb{C})$*-representation ρ is λ-regular, then $C_*(M_K;\widetilde{\mathfrak{sl}}_2(\mathbb{C})_\rho)$ is acyclic.*

Theorem 2.4.3 (Kirk–Livingston [11], Kitano [12])**.** *Let $\mathcal{B}$ be a basis of $\mathfrak{sl}_2(\mathbb{C})$. If $C_*(M_K;\widetilde{\mathfrak{sl}}_2(\mathbb{C})_\rho)$ is acyclic, then the torsion* $\mathrm{Tor}(C_*(M_K;\widetilde{\mathfrak{sl}}_2(\mathbb{C})_\rho), \mathbf{c}^*_{\mathcal{B}})$ *coincides with the twisted Alexander invariant of $\pi_1(M_K)$ and $Ad\circ\rho$.*

The twisted Alexander invariant is given by using Fox differentials. We will review it in the next section.

3. The non-acyclic Reidemeister torsion at bifurcation points

In this section, we will see that the limit of the Reidemeister torsion $\mathbb{T}^K_\lambda$ is given by the differential coefficient of the acyclic Reidemeister torsion $\Delta_K(e^{2z})/(e^z - e^{-z})$ at bifurcation points corresponding to simple roots of the Alexander polynomial of K. Here $\Delta_K(t)$ is normalized, i.e., $\Delta_K(t) = \Delta_K(t^{-1})$ and $\Delta_K(1) = 1$.

3.1. **On the existence of a path of γ-regular characters.** We show that there exists a path of characters of γ-regular representations which converges to a bifurcation point if the function I_γ is not constant on $X^{nab}(M_K)$ near the bifurcation point.

Proposition 3.1.1. *Let z_0 be a complex number such that e^{2z_0} is a simple root of the Alexander polynomial of K and ρ_{z_0} be a reducible non-abelian $\mathrm{SL}_2(\mathbb{C})$-representation whose character is the same as that of the abelian representation φ_{z_0}. Let γ denote a simple closed curve in ∂M_K. If the function I_γ is not constant on the component of $X^{nab}(M_K)$ which contains the character $\chi_{\rho_{z_0}}$, then there exists a neighbourhood V of $\chi_{\rho_{z_0}}$ such that any point of V except for at most finite points is γ-regular.*

We prepare some notions to prove Proposition 3.1.1. Let ρ be an irreducible $\mathrm{SL}_2(\mathbb{C})$-representation of $\pi_1(M_K)$ such that $\rho(\pi_1(\partial M_K))$ contains a non-trivial hyperbolic element of $\mathrm{SL}_2(\mathbb{C})$. Let γ be a simple closed curve in ∂M_K. We can choose a neighbourhood U of χ_ρ such that for any $\rho' \in \pi^{-1}(U)$, the image of the peripheral subgroup $\rho'(\pi_1(\partial M_K))$ also contains a non-trivial hyperbolic element. We can define an analytic function α_γ on U by the following equation:

$$\rho'(\gamma) = A_{\rho'} \begin{pmatrix} e^{\alpha_\gamma} & 0 \\ 0 & e^{-\alpha_\gamma} \end{pmatrix} A_{\rho'}^{-1}$$

where $A_{\rho'} \in \mathrm{SL}_2(\mathbb{C})$ (for details, see Porti [19, Definition 3.19]). Note that this function satisfies the following equation:

$$e^{2\alpha_\gamma(\chi)} - I_\gamma(\chi) e^{\alpha_\gamma(\chi)} + 1 = 0.$$

Proposition 3.26 in Porti [19] gives a criterion about the γ-regularity of ρ.

Lemma 3.1.2 (Consequence of Proposition 3.26 in Porti [19])**.** *Suppose that the dimension of the component containing U is equal to 1. The irreducible representation ρ is γ-regular if and only if $\alpha_\gamma \circ \pi : \pi^{-1}(U) \subset R(\pi_1(M_K), \mathrm{SL}_2(\mathbb{C})) \to \mathbb{C}$ is a submersion at ρ.*

Proposition 3.1.1 follows from Theorem 2.1.4 and Lemma 3.1.2.

Proof of Proposition 3.1.1. We let X_0 denote the component of $X^{nab}(M_K)$ which contains the bifurcation point $\chi_{\rho_{z_0}}$. Theorem 2.1.4 implies that the dimension of X_0 is equal to 1. Since e^{2z_o} is a root of the Alexander polynomial of K, $I_\mu(\chi_{\rho_{z_0}})$ is not equal to ± 2. In particular, $\rho_{z_0}(\mu)$ is a hyperbolic element in $\mathrm{SL}_2(\mathbb{C})$. Thus the subgroup $\rho_{z_0}(\pi_1(\partial M_K))$ consists of hyperbolic elements.

By continuity, we can take a neighbourhood U of $\chi_{\rho_{z_0}}$ in X_0 such that, for every $\chi \in U$, $I_\mu(\chi) \neq \pm 2$. Let V be a compact neighbourhood of $\chi_{\rho_{z_0}}$ in U. Since α_γ is analytic and I_γ is not constant in V, there exist only finite characters where the derivation of α_γ vanishes. Hence, by Lemma 3.1.2, there are only a finite number of characters in V which are not γ-regular. □

Corollary 3.1.3. *If the function I_λ is not constant near $\chi_{\rho_{z_0}}$ on $X^{nab}(M_K)$, then there exists a path of λ-regular characters which converges to $\chi_{\rho_{z_0}}$.*

3.2. **Limits of the non-acyclic Reidemeister torsion for knots at bifurcation points.** If the function I_λ is not constant near a bifurcation point corresponding to a simple root of $\Delta_K(t)$, then there exists a path of λ-regular characters, converging to the bifurcation point. We can consider the limit of the Reidemeister torsion $\mathbb{T}^K_\lambda$ along this path. This limit is obtained from the differential coefficient of $\Delta_K(e^{2z})/(e^z - e^{-z})$ as follows.

Theorem 3.2.1. *Let z_0 be a complex number such that e^{2z_0} is a simple root of the Alexander polynomial $\Delta_K(t)$ of K. Let ρ_{z_0} denote the reducible non-abelian $\mathrm{SL}_2(\mathbb{C})$-representation whose character is the same as one of φ_{z_0}. If the function I_λ is not constant near $\chi_{\rho_{z_0}}$ on $X^{nab}(M_K)$, then the limit of the Reidemeister torsion $\mathbb{T}^K_\lambda$ is expressed as*

$$\lim_{\chi_\rho \to \chi_{\rho_{z_0}}} \mathbb{T}^K_\lambda(\rho) = \varepsilon \cdot \left(\frac{1}{2} \frac{d}{dz} \left(\frac{\Delta_K(e^{2z})}{e^z - e^{-z}} \right) \Big|_{z=z_0} \right)^2 \tag{6}$$

where $\varepsilon \in \{\pm 1\}$.

The function $\Delta_K(e^{2z})/(e^z - e^{-z})$ is regarded as the Reidemeister torsion for the abelian representation ψ_z by Theorem 2.4.1. This relation shows that the Reidemeister torsion for the non-abelian representation ρ_{z_0} is determined by the Reidemeister torsion for the abelian representation ψ_{z_0}.

3.3. **Proof of Theorem 3.2.1.** To prove this theorem, we describe the Reidemeister torsion $\mathbb{T}^K_\lambda(\chi_\rho)$ as the differential coefficient of the sign–determined Reidemeister torsion of $C_*(M_K; \widetilde{\mathfrak{sl}_2}(\mathbb{C})_\rho)$ as follows. (see Theorem 3.1.2 in Yamaguchi [23].)

$$\mathbb{T}^K_\lambda(\chi_\rho) = -\lim_{t \to 1} \frac{\mathcal{T}(M_K; \widetilde{\mathfrak{sl}_2}(\mathbb{C})_\rho, \mathfrak{o})}{t-1}.$$

where we write $\mathcal{T}(M_K; \widetilde{\mathfrak{sl}_2}(\mathbb{C})_\rho, \mathfrak{o})$ instead of $\mathrm{TOR}(M_K; \widetilde{\mathfrak{sl}_2}(\mathbb{C})_\rho, \emptyset, \mathfrak{o})$ for simplicity. We want to know the following limit;

$$\lim_{\rho \to \rho_{z_0}} \left(\lim_{t \to 1} \frac{\mathcal{T}(M_K; \widetilde{\mathfrak{sl}_2}(\mathbb{C})_\rho, \mathfrak{o})}{t-1} \right).$$

Here we take the limit along a path of λ-regular representations, converging to the reducible representation ρ_{z_0}. We investigate the behavior of $\frac{\mathcal{T}(M_K; \widetilde{\mathfrak{sl}_2}(\mathbb{C})_\rho, \mathfrak{o})}{t-1}$ at $\rho = \rho_{z_0}$ and $t = 1$. Since the numerator is regarded as the sign–determined twisted Alexander invariant for K and $Ad \circ \rho$ (for the details, see Kirk–Livingston [11], Kitano [12] and Yamaguchi [23]), it is described more explicitly as follows. Suppose that the group of K has the following presentation:

$$\pi_1(M_K) = \langle x_1, \ldots, x_k \,|\, r_1, \ldots, r_{k-1} \rangle.$$

Since $\alpha : \pi_1(M_K) \to \mathbb{Z}$ is surjective, by interchange columns if necessary, we can assume that $\alpha(x_1) \neq 1$. Then we have that $\det \Phi(x_1 - 1) \neq 1$. By Proposition 2.4.2 and Theorem 2.4.3, if a representation ρ of $\pi_1(M_K)$ is λ-regular, then the chain complex $C_*(M_K; \widetilde{\mathfrak{sl}_2}(\mathbb{C})_\rho)$ is acyclic and its torsion $\mathcal{T}(M_K; \widetilde{\mathfrak{sl}_2}(\mathbb{C})_\rho, \mathfrak{o})$ is well–defined and given by

$$\tau_0 \cdot t^m \frac{\det A^1_{K, Ad\circ\rho}}{\det \Phi(x_1 - 1)}, \tag{7}$$

where m is some integer, the symbol Φ stands for the tensor product homomorphism

$$\alpha \otimes Ad \circ \rho : \mathbb{Z}[\pi_1(M_K)] \to M_3(\mathbb{C}[t, t^{-1}])$$

with respect to a basis of $\mathfrak{sl}_2(\mathbb{C})$ and $A^1_{K, Ad\circ\rho}$ denotes the following $3(k-1) \times 3(k-1)$ matrix over $\mathbb{C}[t, t^{-1}]$:

$$A^1_{K, Ad\circ\rho} = \begin{pmatrix} \Phi(\frac{\partial r_1}{\partial x_2}) & \ldots & \Phi(\frac{\partial r_{k-1}}{\partial x_2}) \\ \vdots & \ddots & \vdots \\ \Phi(\frac{\partial r_1}{\partial x_k}) & \ldots & \Phi(\frac{\partial r_{k-1}}{\partial x_k}) \end{pmatrix}.$$

This rational function is the twisted Alexander invariant defined by Wada [22]. He has shown that the twisted Alexander invariant does not depend on the presentation

of the group. (Theorem 1 in Wada [22]) By the Euclidean algorithm, we can choose the following presentation for the knot group $\pi_1(M_K)$.

Lemma 3.3.1 (Lemma 2.1 in Heusener–Porti–Suárez [10])**.** *If necessary, we can replace the presentation of* $\pi_1(M_K)$ *by* $\langle x'_1, \ldots, x'_k \,|\, r'_1, \ldots, r'_{k-1}\rangle$ *such that* $\alpha(x'_i) = t$ *for all* i*.*

Remark 3.3.2. The chosen presentation is not required to be a Wirtinger presentation in the case of a knot in S^3.

Therefore we can assume from the beginning that $\pi_1(M_K)$ has the presentation:

$$\langle x_1, \ldots, x_k \,|\, r_1, \ldots, r_{k-1}\rangle$$

such that $\alpha(x_i) = t$ for all i.

The rational function (7) is expressed as

$$\tau_0 \cdot \frac{\det A^1_{K,Ad\circ\rho}}{\det \Phi(x_1 - 1)} = \frac{\det A^1_{K,Ad\circ\rho}}{(t-1)(t^2 - \mathrm{Tr}\,(\rho(x_1^2))t + 1)}.$$

Therefore the torsion $\mathcal{T}(M_K, \widetilde{\mathfrak{sl}_2}(\mathbb{C})_\rho, \mathfrak{o})/(t-1)$ is equal to

$$\tau_0 \cdot \frac{\det A^1_{K,Ad\circ\rho}}{(t-1)^2(t^2 - \mathrm{Tr}\,(\rho(x_1^2))t + 1)}$$

up to a factor t^m. Since we suppose that ρ is λ-regular, we know that $(t-1)^2$ divides $\det A^1_{K,Ad\circ\rho}$ (See Section 3.3 in Yamaguchi [23]).

Let $G_\rho(t)$ denote the rational function $(\det A^1_{K,Ad\circ\rho})/(t-1)^2$. We will consider the following two functions $t^2 - \mathrm{Tr}\,(\rho(x_1^2))t + 1$ and $G_\rho(t)$ at $\rho = \rho_{z_0}$ and $t = 1$.

Lemma 3.3.3. *The function* $t^2 - \mathrm{Tr}\,(\rho_{z_0}(x_1^2))t + 1$ *is smooth and non-zero at* $\rho = \rho_{z_0}$ *and* $t = 1$*.*

Proof of Lemma 3.3.3. The function $t^2 - \mathrm{Tr}\,(\rho(x_1^2))t + 1$ depends on ρ smoothly. We look for the value of $t^2 - \mathrm{Tr}\,(\rho(x_1^2))t + 1$ at $\rho = \rho_{z_0}$. By the assumption that ρ_{z_0} has the same character as φ_{z_0}, we have that $\mathrm{Tr}\,(\rho_{z_0}(x_1^2)) = e^{2z_0} + e^{-2z_0}$. Since e^{2z_0} is a simple root of the Alexander polynomial $\Delta_K(t)$ of K and $\Delta_K(1) = 1$, the complex number e^{2z_0} is not equal to 1. Hence if we substitute $t = 1$ into the polynomial $t^2 - \mathrm{Tr}\,(\rho_{z_0}(x_1^2))t + 1$, then its value $2 - (e^{2z_0} + e^{-2z_0})$ is not zero. □

The following proposition plays an important role when we consider the function $G_\rho(t)$ and prove Theorem 3.2.1.

Proposition 3.3.4. *The chain complex* $C_*(M_K; \widetilde{\mathfrak{sl}_2}(\mathbb{C})_{\rho_{z_0}})$ *is acyclic. Moreover the Reidemeister torsion* $\mathcal{T}(M_K; \widetilde{\mathfrak{sl}_2}(\mathbb{C})_{\rho_{z_0}}, \mathfrak{o})$ *is given by*

$$\tau_0 \cdot \epsilon t^m \cdot \frac{\Delta_K(t)\Delta_K(te^{2z_0})\Delta_K(te^{-2z_0})}{(t-1)(t^2 - \mathrm{Tr}\,(\rho_{z_0}(x_1^2))t + 1)} \tag{8}$$

where $\epsilon \in \{\pm 1\}$*,* $m \in \mathbb{Z}$ *and* $\Delta_K(t)$ *is the normalized Alexander polynomial of* K*.*

Proof of Proposition 3.3.4. It is enough to prove the following claims:

- $\det(\Phi(x_1) - 1)$ is not zero;
- $\det A^1_{K,Ad\circ\rho_{z_0}}$ is expressed by using the product of the three Alexander polynomials which appear in the numerator of the fraction in Eq. (8);
- $C_*(M_K; \widetilde{\mathfrak{sl}_2}(\mathbb{C})_{\rho_{z_0}})$ is acyclic and its Reidemeister torsion is given as above.

We have seen that $\det(\Phi(x_1 - 1))$ is not zero. We consider $\det A^1_{K,Ad\circ\rho_{z_0}}$. Since the $\mathrm{SL}_2(\mathbb{C})$-representation ρ_{z_0} has the same character as φ_{z_0} and $\alpha(x_i) = \alpha(\mu)$ for all i, we have that

$$\mathrm{Tr}\,(\rho_{z_0}(x_1)) = \cdots = \mathrm{Tr}\,(\rho_{z_0}(x_k)) = \mathrm{Tr}\,(\rho_{z_0}(\mu)).$$

Furthermore ρ_{z_0} is reducible, then we can assume that

$$\rho_{z_0}(x_i) = \begin{pmatrix} e^{z_0} & \alpha_i \\ 0 & e^{-z_0} \end{pmatrix}$$

by taking conjugation, where α_i is a complex number (Remark 2.1.2). We take an ordered basis $\{E, H, F\}$ of $\mathfrak{sl}_2(\mathbb{C})$ as follows:

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Under this basis, for each x_i, the representation matrix of $Ad(\rho_{z_0}(x_i))$ is given by

$$Ad(\rho_{z_0}(x_i)^{-1}) = \begin{pmatrix} e^{-2z_0} & 2\alpha_i e^{-z_0} & -\alpha_i^2 \\ 0 & 1 & -\alpha_i e^{z_0} \\ 0 & 0 & e^{2z_0} \end{pmatrix}.$$

Note that each $\Phi(\frac{\partial r_i}{\partial x_j})$ is an upper triangular matrix for any i and j.

We express the matrix $\Phi(\frac{\partial r_i}{\partial x_j})$ $(1 \le i \le k-1, 2 \le j \le k)$ by using the following matrix:

$$\begin{pmatrix} a_{ij} & * & * \\ 0 & b_{ij} & * \\ 0 & 0 & c_{ij} \end{pmatrix}.$$

Claim 3.3.5.

$$\det A^1_{K,Ad\circ\rho_{z_0}} = \epsilon t^m \Delta_K(t)\Delta_K(te^{2z_0})\Delta_K(te^{-2z_0})$$

where $\Delta_K(t)$ is the normalized Alexander polynomial of K, $\epsilon \in \{\pm 1\}$ and $m \in \mathbb{Z}$.

Proof of Claim 3.3.5.

$$\det A^1_{K,Ad\circ\rho_{z_0}} = \begin{vmatrix} a_{12} & * & * & a_{22} & * & * & \cdots \\ & b_{12} & * & & b_{22} & * & \vdots \\ & & c_{12} & & & c_{22} & \cdots \\ a_{13} & * & * & a_{23} & * & * & \cdots \\ & b_{13} & * & & b_{23} & * & \vdots \\ & & c_{13} & & & c_{23} & \cdots \\ & \vdots & & & \vdots & & \ddots \end{vmatrix}$$

$$= \begin{vmatrix} A & * & * \\ & B & * \\ & & C \end{vmatrix}.$$

Here A, B and C respectively denote the small matrices $(a_{ij})_{i,j}, (b_{ij})_{i,j}$ and $(c_{ij})_{i,j}$ $(1 \le i \le k-1, 2 \le j \le k)$.

From the equation $\alpha(x_1) = t$ and the calculation of the Alexander polynomial using Fox differentials (Chapter 9 in Burde and Zieschang [2]), we can see that there exist some integer n' and $\epsilon \in \{\pm 1\}$ such that

$$\det A = \epsilon(e^{-2z_0}t)^{n'}\Delta_K(te^{-2z_0}),$$
$$\det B = \epsilon t^{n'}\Delta_K(t),$$
$$\det C = \epsilon(e^{2z_0}t)^{n'}\Delta_K(te^{2z_0}).$$

Therefore we have that

$$\det A^1_{K,Ad\circ\rho_{z_0}} = \epsilon t^{3n'}\Delta_K(t)\Delta_K(te^{2z_0})\Delta_K(te^{-2z_0}).$$

(Claim 3.3.5) □

Hence $C_*(M_K; \widetilde{\mathfrak{sl}}_2(\mathbb{C})_{\rho_{z_0}})$ is acyclic. Furthermore we can see that there exists some integer m such that the sign–determined Reidemeister torsion of $C_*(M_K; \widetilde{\mathfrak{sl}}_2(\mathbb{C})_{\rho_{z_0}})$ is expressed as

$$\mathcal{T}(M_K; \widetilde{\mathfrak{sl}}_2(\mathbb{C})_{\rho_{z_0}}, \mathfrak{o}) = \tau_0 \cdot \epsilon t^m \cdot \frac{\Delta_K(t)\Delta_K(te^{2z_0})\Delta_K(te^{-2z_0})}{(t-1)(t^2 - \mathrm{Tr}\,(\rho_{z_0}(x_1^2))t + 1)}.$$

$\square$

Now we consider the function $G_\rho(t)$ at $\rho = \rho_{z_0}$ and $t = 1$.

Lemma 3.3.6. *The rational function $G_\rho(t)$ is smooth and non-zero at $\rho = \rho_{z_0}$ and $t = 1$.*

Proof of Lemma 3.3.6. Since e^{2z_0} is a simple root of $\Delta_K(t)$ and $\Delta_K(t)$ is symmetric for t, i.e., $\Delta_K(t) = \Delta_K(t^{-1})$, the complex number e^{-2z_0} is also a simple root of $\Delta_K(t)$. Hence the numerator $\Delta_K(t)\Delta_K(te^{2z_0})\Delta_K(te^{-2z_0})$ of Eq. (8) has the second order zero at $t = 1$. Therefore the function $\det A^1_{K,Ad\circ\rho}$ can be divided by $(t-1)^2$ at $\rho = \rho_{z_0}$. We can define $G_{\rho_{z_0}}(t) \in \mathbb{C}[t, t^{-1}]$. Hence the function $G_\rho(t)$ changes smoothly to $G_{\rho_{z_0}}(t)$ and there exists non-zero limit of $G_\rho(t)$ at $\rho = \rho_{z_0}$ and $t = 1$. $\square$

Now, we are ready to calculate the limit of the Reidemeister torsion $\mathbb{T}^K_\lambda$ by using Proposition 3.3.4.

Proof of Themorem 3.2.1. By Lemmas 3.3.3 and 3.3.6, we see that the limit of the rational function $\mathcal{T}(M_K; \widetilde{\mathfrak{sl}}_2(\mathbb{C})_\rho, \mathfrak{o})/(t-1)$ at $\rho = \rho_{z_0}$ and $t = 1$ exists. Moreover when we express the rational function $\mathcal{T}(M_K; \widetilde{\mathfrak{sl}}_2(\mathbb{C})_\rho, \mathfrak{o})/(t-1)$ as $G_\rho(t)/(t^2 - \mathrm{Tr}\,(\rho(x_1^2))+1)$, both of the numerator $G_\rho(t)$ and the denominator $t^2 - \mathrm{Tr}\,(\rho(x_1^2))+1$ are smooth and non-zero near $\rho = \rho_{z_0}$ and $t = 1$. Hence we can change the order of taking limits. By interchanging the limit of t and that of ρ and by Proposition 3.3.4, the limit of $\mathbb{T}^K_\lambda$ is calculated as follows.

$$\begin{aligned}
\lim_{\rho\to\rho_{z_0}} \mathbb{T}^K_\lambda(\rho) &= -\lim_{\rho\to\rho_{z_0}} \left(\lim_{t\to 1} \frac{\mathcal{T}(M_K; \widetilde{\mathfrak{sl}}_2(\mathbb{C})_\rho, \mathfrak{o})}{t-1} \right) \\
&= -\lim_{t\to 1} \frac{\mathcal{T}(M_K; \widetilde{\mathfrak{sl}}_2(\mathbb{C})_{\rho_{z_0}}, \mathfrak{o})}{t-1} \\
&= \lim_{t\to 1} \left\{ \frac{\Delta_K(te^{2z_0})\Delta_K(te^{-2z_0})}{(t-1)^2} \cdot \frac{-\epsilon\tau_0 t^m \Delta_K(t)}{t^2 - \mathrm{Tr}\,(\rho_{z_0}(x_1^2))t + 1} \right\}
\end{aligned}$$

where $\epsilon \in \{\pm 1\}$ and $m \in \mathbb{Z}$.

Since $\Delta_K(1) = 1$ and e^{2z_0} and e^{-2z_0} are simple roots of $\Delta_K(t)$, we have

$$\begin{aligned}
&\lim_{t\to 1} \left\{ \frac{\Delta_K(te^{2z_0})\Delta_K(te^{-2z_0})}{(t-1)^2} \cdot \frac{-\epsilon\tau_0 t^m \Delta_K(t)}{t^2 - \mathrm{Tr}\,(\rho_{z_0}(x_1^2))t + 1} \right\} \\
&= \frac{-\epsilon\tau_0}{2 - (e^{2z_0} + e^{-2z_0})} \cdot \lim_{t\to 1} \left\{ \frac{\Delta_K(te^{2z_0})}{t-1} \cdot \frac{\Delta_K(te^{-2z_0})}{t-1} \right\} \\
&= \frac{-\epsilon\tau_0 \Delta'_K(e^{2z_0})\Delta'_K(e^{-2z_0})}{2 - (e^{2z_0} + e^{-2z_0})}. \qquad (9)
\end{aligned}$$

It follows from the symmetry of $\Delta_K(t)$ that

$$\Delta'_K(e^{-2z_0}) = -\Delta'_K(e^{2z_0})e^{4z_0}. \qquad (10)$$

If we substitute Equation (10) into Equation (9), then we obtain:

$$\lim_{\rho\to\rho_{z_0}} \mathbb{T}^K_\lambda(\rho) = -\tau_0 \cdot \epsilon \cdot \frac{(\Delta'_K(e^{2z_0})e^{2z_0})^2}{(e^{2z_0} + e^{-2z_0}) - 2}.$$

On the other hand, the right hand side of Equation (6) is given by a direct calculation:

$$\left(\frac{1}{2}\,\frac{d}{dz}\left(\frac{\Delta_K(e^{2z})}{e^z - e^{-z}}\right)\bigg|_{z=z_0}\right)^2 = \left(\frac{\Delta'_K(e^{2z_0})e^{2z_0}}{e^{z_0} - e^{-z_0}}\right)^2$$
$$= \frac{(\Delta'_K(e^{2z_0})e^{2z_0})^2}{e^{2z_0} + e^{-2z_0} - 2}.$$

Therefore we have

$$\lim_{\rho \to \rho_{z_0}} \mathbb{T}^K_\lambda(\rho) = \varepsilon \cdot \left(\frac{1}{2}\,\frac{d}{dz}\left(\frac{\Delta_K(e^{2z})}{e^z - e^{-z}}\right)\bigg|_{z=z_0}\right)^2$$

where $\varepsilon = -\tau_0 \cdot \epsilon$, which completes the proof. □

4. On the reference generators of the $\mathfrak{sl}_2(\mathbb{C})_\rho$-homology groups at a bifurcation point

We consider the reference generators of $H_*(M_K; \mathfrak{sl}_2(\mathbb{C})_\rho)$ in this section. By Proposition 3.1.1, the reference generators $\{h^\rho_{(1)}(\lambda), h^\rho_{(2)}\}$ of $H_*(M_K; \mathfrak{sl}_2(\mathbb{C})_\rho)$ exist for any irreducible representation ρ sufficiently near the reducible non-abelian representation ρ_{z_0} when I_λ is not constant on the component of $X^{nab}(M_K)$, containing the bifurcation point $\chi_{\rho_{z_0}}$. Here the representation ρ_{z_0} corresponds to a simple root of the Alexander polynomial of K. We will show that they can be extended to the generator of $H_*(M_K; \mathfrak{sl}_2(\mathbb{C})_{\rho_{z_0}})$. In this section, we assume that the regular function I_λ is not constant on the component containing the bifurcation point $\chi_{\rho_{z_0}}$ in $X^{nab}(M_K)$.

4.1. On the generator of the second $\mathfrak{sl}_2(\mathbb{C})_\rho$-twisted homology group at a bifurcation point.

From the following results of Heusener, Porti and Suárez [10] we know the dimensions of $H_*(M_K; \mathfrak{sl}_2(\mathbb{C})_{\rho_{z_0}})$ and the basis of $H_2(M_K; \mathfrak{sl}_2(\mathbb{C})_{\rho_{z_0}})$.

Lemma 4.1.1 (Lemma 4.1 of Heusener–Porti–Suárez [10])**.** *Let M be a connected, compact, orientable, irreducible 3-manifold such that ∂M is a torus and the first Betti number is one.*

Let $\rho : \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$ be a non-abelian representation such that $\rho(\pi_1(\partial M))$ contains a non-parabolic element. We let i denote the inclusion $\partial M \hookrightarrow M$ and $Z^1(M; \mathfrak{sl}_2(\mathbb{C})_\rho)$ denote the set of twisted cocycles of M with coefficients in $\mathfrak{sl}_2(\mathbb{C})_\rho$. If $\dim_\mathbb{C} Z^1(M; \mathfrak{sl}_2(\mathbb{C})_\rho) = 4$, then we have an injection

$$i^* : H^1(M; \mathfrak{sl}_2(\mathbb{C})_\rho) \to H^1(\partial M; \mathfrak{sl}_2(\mathbb{C})_\rho)$$

and an isomorphism $i^* : H^2(M; \mathfrak{sl}_2(\mathbb{C})_\rho) \to H^2(\partial M; \mathfrak{sl}_2(\mathbb{C})_\rho)$.

We apply this lemma to the knot exterior M_K and ρ_{z_0}. It follows from Proposition 4.4 of Heusener–Porti–Suárez [10] that $\dim_\mathbb{C} Z^1(M_K; \mathfrak{sl}_2(\mathbb{C})_{\rho_{z_0}}) = 4$. Since $\mathrm{Tr}\,(\rho_{z_0}(\mu)) = e^{z_0} + e^{-z_0} \neq \pm 2$, we see that $\rho_{z_0}(\pi_1(\partial M_K))$ contains a non-parabolic element.

Therefore we have that:

- $H_0(M_K; \mathfrak{sl}_2(\mathbb{C})_{\rho_{z_0}}) = 0$;
- the induced homomorphism $i_* : H_1(\partial M_K; \mathfrak{sl}_2(\mathbb{C})_{\rho_{z_0}}) \to H_1(M_K; \mathfrak{sl}_2(\mathbb{C})_{\rho_{z_0}})$ is surjective;
- the induced homomorphism $i_* : H_2(\partial M_K; \mathfrak{sl}_2(\mathbb{C})_{\rho_{z_0}}) \to H_2(M_K; \mathfrak{sl}_2(\mathbb{C})_{\rho_{z_0}})$ is an isomorphism.

Note that $\dim_\mathbb{C} H_2(\partial M_K; \mathfrak{sl}_2(\mathbb{C})_{\rho_{z_0}})$ is equal to 1 since the restriction of ρ_{z_0} to $\pi_1(\partial M_K)$ is non-trivial.

Proposition 4.1.2. *The chain $i_*(P^{\rho_{z_0}} \otimes \widetilde{\partial M_K})$ determines a basis of the homology group $H_2(M_K; \mathfrak{sl}_2(\mathbb{C})_{\rho_{z_0}})$. Here $P^{\rho_{z_0}}$ is a vector in $\mathfrak{sl}_2(\mathbb{C})$ such that $Ad_{\rho_{z_0}(\gamma)}(P^{\rho_{z_0}}) = P^{\rho_{z_0}}$ for all $\gamma \in \pi_1(\partial M_K)$.*

Proof of Proposition 4.1.2. By calculations, we see that $P^{\rho_{z_0}} \otimes \widetilde{\partial M_K}$ is a cycle in $C_2(\partial M_K; \mathfrak{sl}_2(\mathbb{C})_{\rho_{z_0}})$ and it determines a non-zero element of $H_2(M_K; \mathfrak{sl}_2(\mathbb{C})_{\rho_{z_0}})$ (see Porti [19, Proposition 3.18]).

Since $[P^{\rho_{z_0}} \otimes \widetilde{\partial M_K}]$ is a generator of $H_2(\partial M_K; \mathfrak{sl}_2(\mathbb{C})_{\rho_{z_0}})$, we can take $i_*([P^{\rho_{z_0}} \otimes \widetilde{\partial M_K}])$ as a generator of $H_2(M_K; \mathfrak{sl}_2(\mathbb{C})_{\rho_{z_0}})$. $\square$

4.2. **On the generator of the first $\mathfrak{sl}_2(\mathbb{C})_\rho$-twisted homology group at a bifurcation point.** As ∂M_K is a two–dimensional torus, it follows from calculations that $H_1(\partial M_K; \mathfrak{sl}_2(\mathbb{C})_{\rho_{z_0}})$ is generated by $[P^{\rho_{z_0}} \otimes \widetilde{\mu}]$ and $[P^{\rho_{z_0}} \otimes \widetilde{\lambda}]$ (see Porti [19, Proposition 3.18]). The problem lies in whether $i_*([P^{\rho_{z_0}} \otimes \widetilde{\lambda}])$ is zero or not in $H_1(M_K; \mathfrak{sl}_2(\mathbb{C})_{\rho_{z_0}})$. We shall show that $i_*([P^{\rho_{z_0}} \otimes \widetilde{\lambda}])$ is a non-zero class in $H_1(M_K; \mathfrak{sl}_2(\mathbb{C})_{\rho_{z_0}})$. This follows from the fact that the limit of the Reidemeister torsion $\mathbb{T}^K_\lambda$ is not zero. Together with Proposition 4.1.2, the following proposition holds.

Proposition 4.2.1. *Let z_0 be a complex number such that e^{2z_0} is a simple root of the Alexander polynomial of K. Let ρ_{z_0} denote the reducible non-abelian $\mathrm{SL}_2(\mathbb{C})$-representation whose character is the same as that of φ_{z_0}. If I_λ is not constant near the bifurcation point $\chi_{\rho_{z_o}}$, then the reference generators $h^\rho_{(1)}(\lambda)$ and $h^\rho_{(2)}$ can be extended in $H_*(M_K; \mathfrak{sl}_2(\mathbb{C})_{\rho_{z_0}})$.*

Proof of Proposition 4.2.1. It is enough to show that $i_*([P^{\rho_{z_0}} \otimes \widetilde{\lambda}])$ is a non-zero class in $H_1(M_K; \mathfrak{sl}_2(\mathbb{C})_{\rho_{z_0}})$. To this purpose, suppose that $i_*([P^{\rho_{z_0}} \otimes \widetilde{\lambda}])$ is zero in $H_1(M_K; \mathfrak{sl}_2(\mathbb{C})_{\rho_{z_0}})$.

By Theorem 2.1.4, it follows that $\dim_{\mathbb{C}} H_1(M_K; \mathfrak{sl}_2(\mathbb{C})_{\rho_{z_0}}) = 1$ and ρ_{z_0} is a smooth point in the $\mathrm{SL}_2(\mathbb{C})$-representation variety of the knot group. By Corollary 3.1.3, there exists a path $\{\rho_s \,|\, s \in \mathbb{C}, |s| < \epsilon\}$ of $\mathrm{SL}_2(\mathbb{C})$-representations such that $\rho_0 = \rho_{z_0}$ and ρ_s is λ-regular at $s \neq 0$. Here ϵ is a small positive real number. The cohomology group $H^1(M_K; \mathfrak{sl}_2(\mathbb{C})_{\rho_s})$ is isomorphic to the Zariski tangent space of $X(M_K)$ at χ_{ρ_s}. We can take a smooth family of generators $\{\xi_s\}_s$ of $H^1(M_K; \mathfrak{sl}_2(\mathbb{C})_{\rho_s})$ associated with the path $\{\rho_s\}_s$. Using the Kronecker pairing between the homology group $H_1(M_K; \mathfrak{sl}_2(\mathbb{C})_{\rho_s})$ and the cohomology group $H^1(M_K; \mathfrak{sl}_2(\mathbb{C})_{\rho_s})$, we have a family $\{\sigma_s\}_s$ of generators of $H_1(M_K; \mathfrak{sl}_2(\mathbb{C})_{\rho_s})$ such that the Kronecker pairing for σ_s and ξ_s does not vanish for each $s \in \mathbb{C}, |s| < \epsilon$.

We define a non-zero complex number $\mathbb{T}^K_\sigma(\rho_s)$ for each s to be

$$\mathbb{T}^K_\sigma(\rho_s) = \mathrm{TOR}(M_K; \mathfrak{sl}_2(\mathbb{C})_{\rho_s}, \{\sigma_s, h^{\rho_s}_{(2)}\}, \mathfrak{o}).$$

where $h^{\rho_s}_{(2)}$ is the reference generator of $H_2(M_K; \mathfrak{sl}_2(\mathbb{C})_{\rho_s})$. This function depends on s smoothly.

Claim 4.2.2. *Let c_s denote the ratio between $h^{\rho_s}_{(1)}(\lambda)$ and σ_s, i.e., $h^{\rho_s}_{(1)}(\lambda) = c_s \cdot \sigma_s$. Then the following equation holds at $s \neq 0$:*

$$\mathbb{T}^K_\lambda(\rho_s) = c_s \cdot \mathbb{T}^K_\sigma(\rho_s).$$

Proof of Claim 4.2.2. This follows from the base change formula for the Reidemeister torsion (see Dubois [6, Formula (5)] and Porti [19, Proposition 0.2]).

$$\begin{aligned}\mathbb{T}^K_\lambda(\rho_s) &= \mathrm{TOR}(M_K; \mathfrak{sl}_2(\mathbb{C})_{\rho_s}, \{h^{\rho_s}_{(1)}(\lambda), h^{\rho_s}_{(2)}\}, \mathfrak{o}) \\ &= \mathrm{TOR}(M_K; \mathfrak{sl}_2(\mathbb{C})_{\rho_s}, \{\sigma_s, h^{\rho_s}_{(2)}\}, \mathfrak{o}) \cdot [h^{\rho_s}_{(1)}(\lambda)/\sigma_s] \\ &= \mathbb{T}^K_\sigma(\rho_s) \cdot c_s.\end{aligned}$$

(Claim 4.2.2) □

The function $\mathbb{T}^K_\lambda(\rho_s)$ also depends on s smoothly. We have known from Theorem 3.2.1 that there exists the non-zero limit of $\mathbb{T}^K_\lambda(\rho_s)$ taking limit s to 0.

On the other hand, the limit of c_s at $s = 0$ is zero by the assumption. The function $\mathbb{T}^K_\sigma(\rho_s)$ does not have a pole at $s = 0$ by the construction. Hence if we take a limit of s to 0, the function $c_s \cdot \mathbb{T}^K_\sigma(\rho_s)$ must be zero. This is a contradiction. Therefore $i_*([P^{\rho_{z_0}} \otimes \widetilde{\lambda}])$ determines a non-zero class in $H_1(M_K; \mathfrak{sl}_2(\mathbb{C})_{\rho_{z_0}})$. □

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